Bimodules associated to vertex operator algebras

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Abstract

Let V be a vertex operator algebra and $m, n \geq 0$. We construct an $A_n(V)$-$A_m(V)$-bimodule $A_{n,m}(V)$ which determines the action of V from the level m subspace to level n subspace of an admissible V-module. We show how to use $A_{n,m}(V)$ to construct naturally admissible V-modules from $A_m(V)$-modules. We also determine the structure of $A_{n,m}(V)$ when V is rational. 2000MSC:17B69

1 Introduction

The representation theory for a vertex operator algebra [B], [FLM] has been studied largely in terms of representation theory for various associative algebras associated to the vertex operator algebra (see [Z], [KW], [DLM2]-[DLM4], [MT], [DZ], [X]). A sequence of associative algebras $A_n(V)$ for $n \geq 0$ was introduced in [DLM3] to deal with the first $n+1$ homogeneous subspaces of an admissible module. These algebras extend and generalize the associative algebra $A(V)$ constructed in [Z]. The main idea of $A_n(V)$ theory is how to use the first few homogeneous subspaces of a module to determine the whole module. From this point of view, the $A_n(V)$ theory is an analogue of the highest weight module theory for semisimple Lie algebras in the field of vertex operator algebra.

Let $M = \bigoplus_{n=0}^{\infty} M(n)$ be an admissible V-module with $M(0) \neq 0$. (cf. [DLM2]).Then each $M(k)$ is an $A_n(V)$-module for $k \leq n$ [DLM3]. On the other hand, given an $A_n(V)$-module U which cannot factor through $A_{n-1}(V)$ one can construct a Verma type admissible V-module $\bar{M}(U)$ such that $\bar{M}(U)(n) = U$. Also V is rational if and only if $A_n(V)$ is semisimple for all n. So the collection of associative algebras $A_n(V)$ determine the representation theory of V in some sense. However, $A_n(V)$ preserves each homogeneous subspace $M(m)$ for $m \leq n$ and cannot map $M(s)$ to $M(t)$ if $s \neq t$. The goals of the present paper are to alleviate this situation.

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Given two nonnegative integers m, n we will construct an $A_n(V)$-$A_m(V)$-bimoulde $A_{n,m}(V)$ with the property that for any $A_m(V)$-module U which cannot factor through $A_{m-1}(V)$ one can associate a Verma type admissible V-module $M(U) = \bigoplus_{k=0}^{\infty} M(U)(n)$ such that $M(U)(n) = A_{n,m}(V) \otimes_{A_m(V)} U$. The action of V on $M(U)(n)$ is determined by a canonical bimodule homomorphism from $A_{p,n}(V) \otimes_{A_n(V)} A_{n,m}(V)$ to $A_{p,m}(V)$. Also, for a given admissible V-module $W = \bigoplus_{k \geq 0} W(k)$ with $W(0) \neq 0$, there is an $A_n(V)$-$A_m(V)$-bimodule homomorphism from $A_{n,m}(V)$ to $\mathrm{Hom}_{\mathbb{C}}(W(m), W(n))$. So the collection of $A_{n,m}(V)$ for all $m, n \in \mathbb{Z}_+$ determine the action of vertex operator algebra V on its admissible module W completely. This, in fact, is our original motivation to define $A_{n,m}(V)$.

If V is a rational vertex operator algebra, then V has only finitely many irreducible admissible V-modules up to isomorphism and each irreducible V-module is ordinary (see [DLM2]). In this case we let $W^1, ..., W^s$ be the inequivalent irreducible admissible V-modules such that $W^i(0) \neq 0$. Then $A_n(V)$ is the direct sum of full matrix algebras $A_n(V) = \bigoplus_{i=1}^{s} \bigoplus_{k=0}^{n} \mathrm{End}_{\mathbb{C}}(W^i(k))$. We show in this paper that if V is rational then

$$A_{n,m}(V) \cong \bigoplus_{l=0}^{\min\{m,n\}} \left(\bigoplus_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(W^i(m-l), W^i(n-l)) \right).$$

The structure of $A_{n,m}(V)$ for general V will be studied in a sequel to this paper.

We have already mentioned the $A_n(V)$ theory. In fact, the Verma type admissible V-module $M(U)$ has been constructed and denoted by $\bar{M}(U)$ in [DLM3] using the idea of induced module in Lie theory. But our work in this paper leads to a strengthening of this old construction. While the old construction in [DLM3] was given as an abstract quotient of certain induced module for certain Lie algebras, the new construction is explicit and each homogeneous subspace $M(U)(n)$ is obvious. In the case that $U = A_m(V)$ we see immediately that $\bigoplus_{n \geq 0} A_{n,m}(V)$ is an admissible V-module. We expect that the study of bimodules $A_{n,m}(V)$ will lead to a proof of some well known conjecture in representation theory.

The result in this paper is also related to some results obtained in [MNT] where the universal enveloping algebra of a vertex operator algebra is used instead of vertex operator algebra itself in this paper.

The paper is organized as follows: In Section 2 we introduce the $A_n(V)$-$A_m(V)$-bimodule $A_{n,m}(V)$ with lot of technical calculations. In Section 3 we discuss the various properties of $A_{n,m}(V)$ such as the $A_n(V)$-$A_m(V)$-bimodule epimorphism from $A_{n,m}(V)$ to $A_{n-1,m-1}(V)$ induced from the identity map on V, isomorphism between $A_{n,m}(V)$ and $A_{m,n}(V)$ and bimodule homomorphism from $A_{n,p}(V) \otimes_{A_p(V)} A_{p,m}(V)$ to $A_{n,m}(V)$. In Section 4 we first give an $A_n(V)$-$A_m(V)$-bimodule homomorphism from $A_{n,m}(V)$ to $\mathrm{Hom}_{\mathbb{C}}(W(m), W(n))$ for any admissible V-module $W = \bigoplus_{k=0}^{\infty} W(k)$. We also show how to construct an admissible V-module from an $A_m(V)$-module which cannot factor through $A_{m-1}(V)$ by using $A_{n,m}(V)$. In addition we show that $A_n(V)$ and $A_{n,n}(V)$ are the same although $A_{n,n}(V)$ as a quotient space of $A_n(V)$ seems much smaller from the definition. The explicit structure of $A_{n,m}(V)$ is determined if V is rational.

We assume that the reader is familiar with the basic knowledge on the representation theory such as of weak modules, admissible modules and (ordinary) modules as presented in [DLM1]-[DLM2] (also see [FLM], [LL]).

2 $A_n(V)$-$A_m(V)$-bimodule $A_{n,m}(V)$

Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra. An associative algebra $A_n(V)$ for any nonnegative integer n has been constructed in [DLM3] to study the representation theory for vertex operator algebras (see below). For $m, n \in \mathbb{Z}_+$, we will construct an $A_n(V)$-$A_m(V)$-bimodule $A_{n,m}(V)$ in this section. It is hard to see in this section why $A_{n,m}(V)$ is so defined and the motivation for defining $A_{n,m}(V)$ comes from the representation theory of V (see Section 4 below).

For homogeneous $u \in V$, $v \in V$ and $m, n, p \in \mathbb{Z}_+$, define the product $*^n_{m,p}$ on V as follows

$$u *^n_{m,p} v = \sum_{i=0}^{p} (-1)^i \binom{m+n-p+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+n-p+i+1}} Y(u,z)v.$$

If $n = p$, we denote $*^n_{m,p}$ by $\bar{*}^n_m$, and if $m = p$, we denote $*^n_{m,p}$ by $*^n_m$, i.e.,

$$u *^n_m v = \sum_{i=0}^{m} (-1)^i \binom{n+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{n+i+1}} Y(u,z)v,$$

$$u \bar{*}^n_m v = \sum_{i=0}^{n} (-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v.$$

The products $u *^n_m v$ and $u \bar{*}^n_m v$ will induce the right $A_m(V)$-module and left $A_n(V)$-module structure on $A_{n,m}(V)$ which will be defined later.

If $m = n$, then $u *^n_m v$ and $u \bar{*}^n_m v$ are equal, and have been defined in [DLM3]. As in [DLM3] we will denote the product by $u *_n v$ in this case.

Let $O'_{n,m}(V)$ be the linear span of all $u \circ^n_m v$ and $L(-1)u + (L(0) + m - n)u$, where for homogeneous $u \in V$ and $v \in V$,

$$u \circ^n_m v = \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{n+m+2}} Y(u,z)v.$$

Again if $m = n$, $u \circ^n_m v$ has been defined in [DLM3] where it was denoted by $u \circ_n v$. Let $O_n(V) = O'_{n,n}(V)$ (see [DLM3]). The following theorem is obtained in [DLM3].

Theorem 2.1. *The $A_n(V)$ is an associative algebra with product $*_n$ with identity $\mathbf{1} + O_n(V)$.*

We will present more results on $A_n(V)$ and its connection with the representation theory of V from [DLM3] later on when necessary. In order to define $A_{n,m}(V)$ we need several lemmas. In the case that $m = n$ most of these lemmas have been proved in [DLM3]. But when $m \neq n$ even if the old proofs given in [DLM3] work but they are much more complicated and need a lot of modifications. Sometimes we need to find totally new proofs.

Lemma 2.2. *For any $u, v \in V$, $u \circ_m^n v$ lies in $O_n(V)\bar{*}_m^n V$.*

Proof: Let $u, v \in V$. Then

$$
\begin{aligned}
&(L(-1)u)\bar{*}_m^n v = \sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+1+m}}{z^{m+i+1}} \frac{d}{dz} Y(u,z)v \\
&= -(wtu+m+1)\sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v \\
&+ \sum_{i=0}^{n}(m+i+1)(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m+1}}{z^{m+i+2}} Y(u,z)v \\
&= -(wtu+m+1)\sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v \\
&+ \sum_{i=0}^{n}(m+i+1)(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v \\
&+ \sum_{i=0}^{n}(m+i+1)(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+2}} Y(u,z)v
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(L(-1)u + L(0)u)\bar{*}_m^n v \\
&= \sum_{i=0}^{n} i(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v \\
&+ \sum_{i=0}^{n}(m+i+1)(-1)^i \binom{m+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+i+2}} Y(u,z)v \\
&= (n+m+1)(-1)^n \binom{n+m}{m} \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{n+m+2}} Y(u,z)v \\
&= (n+m+1)(-1)^n \binom{n+m}{m} u \circ_m^n v.
\end{aligned}
$$

Note that $L(-1)u + L(0)u \in O_n(V)$. The proof is complete. $\square$

The next lemma is motivated by the commutator relation of vertex operators and will relate the two products $u\bar{*}_m^n v$ and $u *_m^n v$.

Lemma 2.3. *If $u, v \in V$, $p_1, p_2, m \in \mathbb{Z}_+$ with $p_1 + p_2 - m \geq 0$, then*

$$
u *_{m,p_1}^{p_1+p_2-m} v - v *_{m,p_2}^{p_1+p_2-m} u - \mathrm{Res}_z (1+z)^{wtu-1+m-p_2} Y(u,z)v \in O'_{p_1+p_2-m,m}(V).
$$

Proof: From the definition of $O'_{p_1+p_2-m,m}(V)$, one can easily verify that

$$
Y(v,z)u \equiv (1+z)^{-wtu-wtv-2m+p_1+p_2} Y(u, \frac{-z}{1+z})v
$$

modulo $O'_{p_1+p_2-m,m}(V)$ (cf. [Z] and [DLM2]). Hence

$$v *_{m,p_2}^{p_1+p_2-m} u = \sum_{i=0}^{p_2}(-1)^i\binom{p_1+i}{i}\mathrm{Res}_z\frac{(1+z)^{wt v+m}}{z^{p_1+i+1}}Y(v,z)u$$

$$\equiv \sum_{i=0}^{p_2}(-1)^i\binom{p_1+i}{i}\mathrm{Res}_z\frac{(1+z)^{wt v+m}}{z^{p_1+i+1}}(1+z)^{-wt u-wt v-2m+p_1+p_2}Y(u,\frac{-z}{1+z})v$$

$$(\mathrm{mod}\ O'_{p_1+p_2-m,m}(V))$$

$$= \sum_{i=0}^{p_2}(-1)^{p_1}\binom{p_1+i}{i}\mathrm{Res}_z\frac{(1+z)^{wt u+i-1+m-p_2}}{z^{p_1+i+1}}Y(u,z)v.$$

Recall the definition of $u *_{m,p_1}^{p_1+p_2-m} v$. Then

$$u *_{m,p_1}^{p_1+p_2-m} v - v *_{m,p_2}^{p_1+p_2-m} u \equiv \mathrm{Res}_z A_{p_1,p_2}(z)(1+z)^{wt u-1+m-p_2}Y(u,z)v,$$

where

$$A_{p_1,p_2}(z) = \sum_{i=0}^{p_1}(-1)^i\binom{p_2+i}{i}\frac{(1+z)^{p_2+1}}{z^{p_2+i+1}} - \sum_{i=0}^{p_2}(-1)^{p_1}\binom{p_1+i}{i}\frac{(1+z)^i}{z^{p_1+i+1}}.$$

The lemma now follows from Proposition 5.1 in the Appendix. $\square$

The proof of the following lemma is fairly standard (cf. [DLM3] and [Z]).

Lemma 2.4. *For homogeneous $u, v \in V$, and integers $k \geq s \geq 0$,*

$$\mathrm{Res}_z\frac{(1+z)^{wt u+m+s}}{z^{n+m+2+k}}Y(u,z)v \in O'_{n,m}(V).$$

Lemma 2.5. *We have $V \bar{*}_m^n O'_{n,m}(V) \subseteq O'_{n,m}(V)$, $O'_{n,m}(V) *_m^n V \subseteq O'_{n,m}(V)$.*

Proof: For homogeneous $u, v \in V$ and $w \in V$,

$$u \bar{*}_m^n (v \circ_m^n w)$$

$$\equiv \sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\mathrm{Res}_{z_1}\frac{(1+z_1)^{wt v+m}}{z_1^{m+i+1}}Y(u,z_1)\mathrm{Res}_{z_2}\frac{(1+z_2)^{wt v+m}}{z_2^{n+m+2}}Y(v,z_2)w$$

$$-\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\mathrm{Res}_{z_2}\frac{(1+z_2)^{wt v+m}}{z_2^{n+m+2}}Y(v,z_2)\mathrm{Res}_{z_1}\frac{(1+z_1)^{wt v+m}}{z_1^{m+i+1}}Y(u,z_1)w$$

$$= \sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{wt u+m}{j}\sum_{k=0}^{\infty}\binom{-m-i-1}{k}$$

$$\cdot\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}\frac{(1+z_2)^{wt u+wt v+2m-j}(z_1-z_2)^{j+k}}{z_2^{2m+n+i+3+k}}Y(Y(u,z_1-z_2)v,z_2)w$$

$$= \sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{wt u+m}{j}\sum_{k=0}^{\infty}\binom{-m-i-1}{k}$$

$$\cdot\mathrm{Res}_{z_2}\frac{(1+z_2)^{wt u+wt v+2m-j}}{z_2^{2m+n+i+3+k}}Y(u_{j+k}v,z_2)w.$$

Note that the weight of $u_{j+k}v$ is $wtu + wtv - j - k - 1$. By Lemma 2.4 we see that $u\bar{*}_m^n(v \circ_m^n w)$ lies in $O'_{n,m}(V)$.

By Lemma 2.3, we have

$$\begin{aligned}
& u\bar{*}_m^n(v \circ_m^n w) - (v \circ_m^n w)*_m^n u \\
\equiv\ & \mathrm{Res}_z(1+z)^{wtu-1}Y(u,z)(v \circ_m^n w) \\
=\ & \mathrm{Res}_{z_1}(1+z_1)^{wtu-1}\mathrm{Res}_{z_2}\frac{(1+z_2)^{wtv+m}}{z_2^{n+m+2}}Y(u,z_1)Y(v,z_2)w \\
\equiv\ & \sum_{j\geq 0}\binom{wtu-1}{j}\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}\frac{(1+z_2)^{wtu+wtv+m-1-j}(z_1-z_2)^j}{z_2^{n+m+2}} \\
& \cdot Y(Y(u,z_1-z_2)v,z_2)w \\
=\ & \sum_{j\geq 0}\binom{wtu-1}{j}\mathrm{Res}_{z_2}\frac{(1+z_2)^{wtu+wtv+m-1-j}}{z_2^{n+m+2}}Y(u_jv,z_2)w \in O'_{n,m}(V)
\end{aligned}$$

which is a vector in $O'_{n,m}(V)$ by the definition of $O'_{n,m}(V)$. As a result, $(v \circ_m^n w)*_m^n u \in O'_{n,m}(V)$.

Next we deal with $L(-1)u + (L(0)+m-n)u \in O'_{n,m}(V)$. As before we assume that u is homogeneous. Then

$$\begin{aligned}
& (L(-1)u + (L(0)+m-n)u)*_m^n v \\
=\ & \sum_{i=0}^m(-1)^i\binom{n+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+m+1}}{z^{n+i+1}}Y(L(-1)u,z)v \\
& +\sum_{i=0}^m(-1)^i\binom{n+i}{i}(wtu+m-n)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+1}}Y(u,z)v \\
=\ & \sum_{i=0}^m(-1)^{i+1}\binom{n+i}{i}(wtu+m+1)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+1}}Y(u,z)v \\
& +\sum_{i=0}^m(-1)^i\binom{n+i}{i}(n+i+1)\mathrm{Res}_z\frac{(1+z)^{wtu+m+1}}{z^{n+i+2}}Y(u,z)v \\
& +\sum_{i=0}^m(-1)^i\binom{n+i}{i}(wtu+m-n)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+1}}Y(u,z)v \\
=\ & \sum_{i=0}^m(-1)^{i+1}\binom{n+i}{i}(n+1)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+1}}Y(u,z)v \\
& +\sum_{i=0}^m(-1)^i\binom{n+i}{i}(n+i+1)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+1}}Y(u,z)v \\
& +\sum_{i=0}^m(-1)^i\binom{n+i}{i}(n+i+1)\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{n+i+2}}Y(u,z)v.
\end{aligned}$$

It is easy to show that the last expression is equal to $(-1)^m(n+m+1)\binom{m+n}{m}u\circ_n v$. By Lemma 2.2, $(L(-1)u+(L(0)+m-n)u)*_m^n v$ belongs to $O'_{n,m}(V)$.

Finally for $v\bar{*}_m^n(L(-1)u+(L(0)+m-n)u)$ we use Lemma 2.3 to yield

$$
\begin{aligned}
& v\bar{*}_m^n(L(-1)u+(L(0)+m-n)u)-(L(-1)u+(L(0)+m-n)u)*_m^n v \\
&= \mathrm{Res}_z(1+z)^{wtv-1}Y(v,z)(L(-1)u+(L(0)+m-n)u) \\
&= \sum_{i\geq 0}\binom{wtv-1}{i}v_iL(-1)u+(wtu+m-n)\sum_{i\geq 0}\binom{wtv-1}{i}v_iu \\
&= L(-1)\sum_{i\geq 0}\binom{wtv-1}{i}v_iu+\sum_{i\geq 0}\binom{wtv-1}{i}iv_{i-1}u \\
&\quad +(wtu+m-n)\sum_{i\geq 0}\binom{wtv-1}{i}v_iu \\
&= L(-1)\sum_{i\geq 0}\binom{wtv-1}{i}v_iu+\sum_{i\geq 0}\binom{wtv-1}{i+1}(i+1)v_iu \\
&\quad +(wtu+m-n)\sum_{i\geq 0}\binom{wtv-1}{i}v_iu \\
&= \sum_{i\geq 0}\binom{wtv-1}{i}(L(-1)+wtv-i-1+wtu+m-n)v_iu \\
&= \sum_{i\geq 0}\binom{wtv-1}{i}(L(-1)v_iu+L(0)v_iu+(m-n)v_iu)
\end{aligned}
$$

which is in $O'_{n,m}(V)$. So $v\bar{*}_m^n(L(-1)u+(L(0)+m-n)u)\in O'_{n,m}(V)$, as desired. □

We should remind the reader that our goal is to construct an $A_n(V)$-$A_m(V)$-bimodule $A_{n,m}(V)$ with the left action $\bar{*}_m^n$ of $A_n(V)$ and the right action $*_m^n$ of $A_m(V)$. The following lemma claims that the left action $\bar{*}_m^n$ and the right action $*_m^n$ commute. On the other hand, we do not need to prove this lemma as a bigger subspace $O_{n,m}(V)$ of V containing $O'_{n,m}(V)$ will be modulo out. In fact, $(a\bar{*}_m^n b)*_m^n c-a\bar{*}_m^n(b*_m^n c)$ is an element of $O_{n,m}(V)$ (see Lemma 2.6 below). But eventually we expect to prove that $O_{m,n}(V)$ and $O'_{n,m}(V)$ are the same although we cannot achieve this in the paper.

Lemma 2.6. *We have $(a\bar{*}_m^n b)*_m^n c-a\bar{*}_m^n(b*_m^n c)$ lies in $O'_{n,m}(V)$ for homogeneous $a,b,c\in V$.*

Proof: The proof of this lemma is similar to that of Theorem 2.4 of [DLM3]. In fact, if $m=n$, the lemma is exactly the associativity of product $*_n$ in $A_n(V)$.

A straightforward calculation using Lemma 2.4 gives:

$$
\begin{aligned}
&(a\bar{*}_m^n b) *_m^n c \\
={}& \sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{wta+m}{j} \\
&\cdot \mathrm{Res}_z\frac{(1+z)^{wta+wtb+2m-j+i}}{z^{n+k+1}}Y(a_{j-m-i-1}b,z)c \\
={}& \sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{wta+m}{j} \\
&\cdot \mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}\frac{(1+z_2)^{wta+wtb+2m-j+i}(z_1-z_2)^{j-m-i-1}}{z_2^{n+k+1}}Y(Y(a,z_1-z_2)b,z_2)c \\
={}& \sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2} \\
&\cdot \frac{(1+z_1)^{wta+m}(1+z_2)^{wtb+m+i}(z_1-z_2)^{-m-i-1}}{z_2^{n+k+1}}Y(Y(a,z_1-z_2)b,z_2)c \\
={}& \sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{-m-i-1}{j} \\
&\cdot \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\frac{(1+z_1)^{wta+m}(1+z_2)^{wtb+m+i}(-z_2)^j}{z_1^{m+i+1+j}z_2^{n+k+1}}Y(a,z_1)Y(b,z_2)c \\
&-\sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j\geq 0}\binom{-m-i-1}{j} \\
&\cdot \mathrm{Res}_{z_2}\mathrm{Res}_{z_1}\frac{(1+z_1)^{wta+m}(1+z_2)^{wtb+m+i}z_1^j}{(-z_2)^{m+i+1+j}z_2^{n+k+1}}Y(b,z_2)Y(a,z_1)c \\
\equiv{}& a\bar{*}_m^n(b *_m^n c)+\sum_{k=0}^{m}(-1)^k\binom{n+k}{k}\sum_{i=0}^{n}(-1)^i\binom{m+i}{i} \\
&\cdot \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\left[\sum_{j=0}^{n-i}(-1)^j\binom{-m-i-1}{j}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{j+l}}{z_1^{j+i}}-\frac{1}{z_1^i}\right] \\
&\cdot \frac{(1+z_1)^{wta+m}(1+z_2)^{wtb+m}}{z_1^{m+1}z_2^{n+k+1}}Y(a,z_1)Y(b,z_2)c.
\end{aligned}
$$

The lemma then follows from Proposition 5.2 in the Appendix. $\square$

In order to construct $A_{n,m}(V)$ we need to introduce more subspaces of V. Let $O''_{n,m}(V)$ be the linear span of $u*_{m,p_3}^n((a*_{p_1,p_2}^{p_3}b)*_{m,p_1}^{p_3}c-a*_{m,p_2}^{p_3}(b*_{m,p_1}^{p_2}c))$, for $a,b,c,u\in V, p_1,p_2,p_3\in \mathbb{Z}_+$, and $O'''_{n,m}(V)=\sum_{p\in\mathbb{Z}_+}(V*_p^n O_p(V))*_{m,p}^n V$. Set

$$O_{n,m}(V)=O'_{n,m}(V)+O''_{n,m}(V)+O'''_{n,m}(V).$$

Lemma 2.7. *For $p_1, p_2, m, n \in \mathbb{Z}_+$, we have $(V *^n_{p_1,p_2} O'_{p_2,p_1}(V)) *^n_{m,p_1} V \subseteq O_{n,m}(V)$.*

Proof: We first prove that $(L(-1)u + (L(0) + p_1 - p_2)u) *^{p_2}_{m,p_1} v \in O_{p_2}(V) *^{p_2}_{m,p_2} V$, for homogeneous $u \in V$ and $v \in V$. In fact,

$$
\begin{aligned}
&(L(-1)u + (L(0) + p_1 - p_2)u) *^{p_2}_{m,p_1} v \\
&= \sum_{i=0}^{p_1} (-1)^i \binom{m+p_2-p_1+i}{i} \mathrm{Res}_z \frac{(1+z)^{wtu+m+1}}{z^{m+p_2-p_1+i+1}} Y(L(-1)u, z)v \\
&\quad + \sum_{i=0}^{p_1} (-1)^i \binom{m+p_2-p_1+i}{i} (wtu + p_1 - p_2) \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+p_2-p_1+i+1}} Y(u, z)v \\
&= \sum_{i=0}^{p_1} (-1)^{i+1} \binom{m+p_2-p_1+i}{i} (wtu + m + 1) \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+p_2-p_1+i+1}} Y(u, z)v \\
&\quad + \sum_{i=0}^{p_1} (-1)^i \binom{m+p_2-p_1+i}{i} (m + p_2 - p_1 + i + 1) \mathrm{Res}_z \frac{(1+z)^{wtu+m+1}}{z^{m+p_2-p_1+i+2}} Y(u, z)v \\
&\quad + \sum_{i=0}^{p_1} (-1)^i \binom{m+p_2-p_1+i}{i} (wtu + p_1 - p_2) \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+p_2-p_1+i+1}} Y(u, z)v \\
&= (-1)^{p_1} \binom{m+p_2}{p_1} (m + p_2 + 1) \mathrm{Res}_z \frac{(1+z)^{wtu+m}}{z^{m+p_2+2}} Y(u, z)v.
\end{aligned}
$$

So by Lemma 2.2, $(L(-1)u + (L(0) + p_1 - p_2)u) *^{p_2}_{m,p_1} v \in O_{p_2}(V) *^{p_2}_{m,p_2} V$. Hence by the definitions of $O'_{n,m}(V)$ and $O_{n,m}(V)$, and Lemma 2.2, we have

$$
\begin{aligned}
&(V *^n_{p_1,p_2} O'_{p_2,p_1}(V)) *^n_{m,p_1} V \subseteq V *^n_{m,p_2} (O'_{p_2,p_1}(V) *^{p_2}_{m,p_1} V) + O_{n,m}(V) \\
&\subseteq V *^n_{m,p_2} ((O_{p_2}(V) *^{p_2}_{p_1,p_2} V) *^{p_2}_{m,p_1} V + O_{p_2}(V) *^{p_2}_{m,p_2} V) + O_{n,m}(V) \\
&\subseteq V *^n_{m,p_2} (O_{p_2}(V) *^{p_2}_{m,p_2} (V *^{p_2}_{m,p_1} V)) + (V *^n_{p_2} O_{p_2}(V)) *^n_{m,p_2} V + O_{n,m}(V) \\
&\subseteq (V *^n_{p_2} O_{p_2}(V)) *^n_{m,p_2} (V *^{p_2}_{m,p_1} V) + O_{n,m}(V) \\
&\subseteq O_{n,m}(V),
\end{aligned}
$$

as required. □

Lemma 2.8. *For any $m, n, p \in \mathbb{Z}_+$, we have $V *^n_{m,p} O_{p,m}(V) \subseteq O_{n,m}(V)$, $O_{n,p}(V) *^n_{m,p} V \subseteq O_{n,m}(V)$. In particular, $V \bar{*}^n_m O_{n,m}(V) \subseteq O_{n,m}(V)$, $O_{n,m}(V) *^n_m V \subseteq O_{n,m}(V)$.*

Proof: By the definition of $O_{n,m}(V)$ and Lemma 2.7, it suffices to prove that

$$V *^n_{m,p} ((V *^p_{p_1} O_{p_1}(V)) *^p_{m,p_1} V + O''_{p,m}(V)) \subseteq O_{n,m}(V) \tag{2.1}$$

and

$$((V *^n_{p_1} O_{p_1}(V)) *^n_{p,p_1} V + O''_{n,p}(V)) *^n_{m,p} V \subseteq O_{n,m}(V) \tag{2.2}$$

for $p_1, p \in \mathbb{Z}_+$.

We first prove (2.1). It is clear that

$$
\begin{aligned}
&V*^n_{m,p}((V*^p_{p_1} O_{p_1}(V))*^p_{m,p_1}V) \subseteq V*^n_{m,p}(V*^p_{m,p_1}(O_{p_1}(V)*^{p_1}_{m,p_1}V)) + O_{n,m}(V)\\
&\subseteq (V*^n_{p_1,p}V)*^n_{m,p_1}(O_{p_1}(V)*^{p_1}_{m,p_1}V) + O_{n,m}(V)\\
&\subseteq ((V*^n_{p_1,p}V)*^n_{p_1}O_{p_1}(V))*^n_{m,p_1}V + O_{n,m}(V) \subseteq O_{n,m}(V).
\end{aligned}
$$

It remains to prove that $V*^n_{m,p}O''_{p,m}(V) \subseteq O_{n,m}(V)$. With $u = \mathbf{1}$ in the definition of $O''_{n,m}(V)$ we have $(a*^n_{p_1,p_2}b)*^n_{m,p_1}c - a*^n_{m,p_2}(b*^{p_2}_{m,p_1}c) \in O_{n,m}(V)$. Thus

$$
\begin{aligned}
&v*^n_{m,p}(u*^p_{m,p_3}((a*^{p_3}_{p_1,p_2}b)*^{p_3}_{m,p_1}c - a*^{p_3}_{m,p_2}(b*^{p_2}_{m,p_1}c)))\\
&\equiv (v*^n_{p_3,p}u)*^n_{m,p_3}((a*^{p_3}_{p_1,p_2}b)*^{p_3}_{m,p_1}c - a*^{p_3}_{m,p_2}(b*^{p_2}_{m,p_1}c))\\
&\equiv 0 \pmod{O_{n,m}(V)}.
\end{aligned}
$$

So (2.1) is true.

For (2.2), it is easy to see that $((V*^n_{p_1}O_{p_1}(V))*^n_{p,p_1}V)*^n_{m,p}V \subseteq (V*^n_{p_1}O_{p_1}(V))*^n_{m,p_1}(V*^{p_1}_{m,p}V) + O_{n,m}(V) \subseteq O_{n,m}(V)$. Let $a,b,c,u,v \in V$ and $p_1,p_2,p_3,p \in \mathbb{Z}_+$, then

$$
\begin{aligned}
&(u*^n_{p,p_3}((a*^{p_3}_{p_1,p_2}b)*^{p_3}_{p,p_1}c - a*^{p_3}_{p,p_2}(b*^{p_2}_{p,p_1}c)))*^n_{m,p}v\\
&\equiv u*^n_{m,p_3}(((a*^{p_3}_{p_1,p_2}b)*^{p_3}_{p,p_1}c - a*^{p_3}_{p,p_2}(b*^{p_2}_{p,p_1}c))*^{p_3}_{m,p}v)\\
&\equiv u*^n_{m,p_3}((a*^{p_3}_{p_1,p_2}b)*^{p_3}_{m,p_1}(c*^{p_1}_{m,p}v)) - u*^n_{m,p_3}(a*^{p_3}_{m,p_2}((b*^{p_2}_{p,p_1}c)*^{p_2}_{m,p}v))\\
&\equiv u*^n_{m,p_3}(a*^{p_3}_{m,p_2}(b*^{p_2}_{m,p_1}(c*^{p_1}_{m,p}v))) - (u*^n_{p_2,p_3}a)*^n_{m,p_2}((b*^{p_2}_{p,p_1}c)*^{p_2}_{m,p}v)\\
&\equiv (u*^n_{p_2,p_3}a)*^n_{m,p_2}(b*^{p_2}_{m,p_1}(c*^{p_1}_{m,p}v)) - (u*^n_{p_2,p_3}a)*^n_{m,p_2}((b*^{p_2}_{p,p_1}c)*^{p_2}_{m,p}v)\\
&\equiv 0 \pmod{O_{n,m}(V)}.
\end{aligned}
$$

The lemma is proved. □

We now define

$$A_{n,m}(V) = V/O_{n,m}(V).$$

The reason for this definition will become clear from the representation theory of V discussed later. The following is the first main theorem in this paper.

Theorem 2.9. *Let V be a vertex operator algebra and m,n nonnegative integers. Then $A_{n,m}(V)$ is an $A_n(V)$-$A_m(V)$-bimodule such that the left and right actions of $A_n(V)$ and $A_m(V)$ are given by $\bar{*}^n_m$ and $*^n_m$.*

Proof: First, both actions are well defined from the definition of $O_{n,m}(V)$ and Lemma 2.8. The left $A_n(V)$-module and right $A_m(V)$-module structures then follow from the fact that $O''_{n,m}(V)$ is a subspace of $O_{n,m}(V)$. The commutativity of two actions proved in Lemma 2.6 asserts that $A_{n,m}(V)$ is an $A_n(V)$-$A_m(V)$-bimodule. □

Remark 2.10. *We will prove in Section 4 that if $m = n$, the $A_{n,n}(V)$ defined here is the same as $A_n(V)$ discussed before. In particular, $O_{n,n}(V)$ and $O'_{n,n}(V)$ coincide. In other words, $O''_{n,n}(V)$, $O'''_{n,n}(V)$ are subspaces of $O'_{n,n}(V)$. We suspect that this is true in general. That is, $O_{n,m}(V)$ and its subspace $O'_{n,m}(V)$ are equal. It seems that this is a very difficult problem and we cannot find a proof for this in this paper.*

3 Properties of $A_{n.m}(V)$

In this section we will discuss some important properties of $A_{n.m}(V)$ such as isomorphism between $A_{n,m}(V)$ and $A_{m,n}(V)$, relations between $A_{n,m}(V)$ and $A_{l,k}(V)$ and tensor products. Some of these properties will be interpreted in terms representation theory in the next section.

First we establish the isomorphism between $A_{n,m}(V)$ and $A_{m,n}(V)$ as $A_n(V)$-$A_m(V)$-bimodules. To achieve this we need to define new actions of $A_n(V)$ and $A_m(V)$ on $A_{m,n}(V)$ so that $A_{m,n}(V)$ becomes an $A_n(V)$-$A_m(V)$-bimodule. Recall from [Z] the linear map $\phi : V \to V$ such that $\phi(v) = e^{L(1)}(-1)^{L(0)}v$ for $v \in V$. Then ϕ induces an anti involution on $A_n(V)$ [DLM3] (also see [Z],[DLM2]). We also use ϕ at the present situation to define an isomorphism between $A_{n,m}(V)$ and $A_{m,n}(V)$.

Lemma 3.1. *For $u, v \in V$ define*

$$u \bar{\cdot}_m^n v = v *_n^m \phi(u), \quad u \cdot_m^n v = \phi(v) \bar{*}_n^m u.$$

Then $A_{m,n}(V)$ becomes an $A_n(V)$-$A_m(V)$-bimodule under the left action $\bar{\cdot}_m^n$ by $A_n(V)$ and the right action $\cdot_m^n$ by $A_m(V)$.

Proof: Since $\phi(O_m(V)) \subset O_m(V)$ for any m (see [DLM3]), we immediately see that both actions are well defined. The rest follows from Theorem 2.9 and the fact that ϕ is an anti involution of $A_m(V)$ for any m. $\square$

Proposition 3.2. *The linear map ϕ induces an isomorphism of $A_n(V)$-$A_m(V)$-bimodules from $A_{n,m}(V)$ to $A_{m,n}(V)$, where the actions of $A_n(V)$ and $A_m(V)$ on $A_{n,m}(V)$ are defined in Theorem 2.9, and the actions on $A_{m,n}(V)$ are defined in Lemma 3.1.*

Proof: We first prove that $\phi(u *_{m,p}^n v) \equiv \phi(v) *_{n,p}^m \phi(u)$ modulo $O'_{m,n}(V)$ for $u, v \in V$ and $p \in \mathbb{Z}_+$. Recall the identities

$$(-1)^{L(0)} Y(u,z)(-1)^{L(0)} = Y((-1)^{L(0)}u, -z)$$

$$e^{L(1)} Y(u,z) e^{-L(1)} = Y(e^{(1-z)L(1)}(1-z)^{-2L(0)}u, \frac{z}{1-z})$$

from [FLM]. We have the following computation with the help from Proposition 5.1 in

Appendix:

$$
\begin{aligned}
&\phi(u*^n_{m,p}v)=\phi\left(\sum_{i=0}^{p}(-1)^i\binom{m+n-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{m+n-p+i+1}}Y(u,z)v\right)\\
&=\sum_{i=0}^{p}(-1)^i\binom{m+n-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{m+n-p+i+1}}e^{L(1)}Y((-1)^{L(0)}u,-z)(-1)^{L(0)}v\\
&=\sum_{i=0}^{p}(-1)^i\binom{m+n-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+m}}{z^{m+n-p+i+1}}\\
&\quad\cdot Y(e^{(1+z)L(1)}(1+z)^{-2L(0)}(-1)^{L(0)}u,\frac{-z}{1+z})e^{L(1)}(-1)^{L(0)}v\\
&=\sum_{i=0}^{p}(-1)^{wtu+m+n-p}\binom{m+n-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+n-p+i-1}}{z^{m+n-p+i+1}}\\
&\quad\cdot Y(e^{(1+z)^{-1}L(1)}u,z)e^{L(1)}(-1)^{L(0)}v\\
&=\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{p}(-1)^{wtu+m+n-p}\binom{m+n-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu+n-p-j+i-1}}{z^{m+n-p+i+1}}\\
&\quad\cdot Y(L(1)^ju,z)e^{L(1)}(-1)^{L(0)}v\\
&=\sum_{j=0}^{\infty}\frac{(-1)^{wtu}}{j!}\sum_{i=0}^{m+n-p}(-1)^i\binom{p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wtu-j+n}}{z^{p+i+1}}Y(L(1)^ju,z)e^{L(1)}(-1)^{L(0)}v\\
&\quad-\sum_{j=0}^{\infty}\frac{(-1)^{wtu}}{j!}\mathrm{Res}_z(1+z)^{wtu-j-1+n-p}Y(L(1)^ju,z)e^{L(1)}(-1)^{L(0)}v\\
&\equiv\phi(v)*^m_{n,p}\phi(u)\ (\mathrm{mod}\ O'_{m,n}(V)),
\end{aligned}
$$

where we have used Proposition 5.1 and Lemma 2.3 in the last two steps. In particular, $\phi(u\bar{*}^n_m v)\equiv\phi(v)*^m_n\phi(u)$ modulo $O'_{n,m}(V)$ and $\phi(u*^n_m v)\equiv\phi(v)\bar{*}^m_n\phi(u)$ modulo $O'_{m,n}(V)$ for $u,v\in V$.

We next prove that $\phi(O_{n,m}(V))\subset O_{m,n}(V)$. Take $v=\mathrm{Res}_z\frac{(1+z)^{wta+m}}{z^{n+m+2}}Y(a,z)b\in O'_{n,m}(V)$. Then

$$
\begin{aligned}
&\phi(v)=\mathrm{Res}_z e^{L(1)}(-1)^{L(0)}\frac{(1+z)^{wta+m}}{z^{n+m+2}}Y(a,z)b\\
&=\mathrm{Res}_z e^{L(1)}\frac{(1+z)^{wta+m}}{z^{n+m+2}}Y(e^{(1+z)L(1)}(1+z)^{-2L(0)}(-1)^{L(0)}a,\frac{-z}{1+z})e^{L(1)}(-1)^{L(0)}b\\
&=\mathrm{Res}_z(-1)^{wta+m+n+1}\frac{(1+z)^{wta+n}}{z^{n+m+2}}Y(e^{\frac{1}{1+z}L(1)}a,z)e^{L(1)}(-1)^{L(0)}b\\
&=\mathrm{Res}_z(-1)^{wta+m+n+1}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{(1+z)^{wta+n-j}}{z^{n+m+2}}Y(L(1)^ja,z)e^{L(1)}(-1)^{L(0)}b\in O'_{m,n}(V).
\end{aligned}
$$

For $u \in V$, we have

$$
\begin{aligned}
&\phi(L(-1)u + (L(0) + m - n)u) \\
&= e^{L(1)}(-1)^{L(0)}\mathrm{Res}_z(Y(\omega, z)u + zY(\omega, z)u) + (m-n)e^{L(1)}(-1)^{L(0)}u \\
&= e^{L(1)}\mathrm{Res}_z(Y(\omega, -z) + zY(\omega, -z))(-1)^{L(0)}u + (m-n)e^{L(1)}(-1)^{L(0)}u \\
&= \mathrm{Res}_z(1+z)Y(e^{(1+z)L(1)}(1+z)^{-2L(0)}(-1)^{L(0)}\omega, \frac{-z}{1+z})e^{L(1)}(-1)^{L(0)}u \\
&\quad +(m-n)e^{L(1)}(-1)^{L(0)}u \\
&= \mathrm{Res}_z(-(1+z)^2 + z(1+z))Y(e^{(1+z)^{-1}L(1)}\omega, z)e^{L(1)}(-1)^{L(0)}u \\
&\quad +(m-n)e^{L(1)}(-1)^{L(0)}u \\
&= -(L(-1) + L(0))e^{L(1)}(-1)^{L(0)}u - (n-m)e^{L(1)}(-1)^{L(0)}u \in O'_{m,n}(V).
\end{aligned}
$$

So $\phi(O'_{n,m}(V)) \subset O'_{m,n}(V)$.

Let $u \in O'_{n,p}(V)$, $w \in O'_{p,m}(V)$, $v \in V$. Then $\phi(u) \in O'_{p,n}(V)$ and $\phi(w) \in O'_{m,p}(V)$. From the proof above, we have $\phi(u *_{m,p}^n v) \equiv \phi(v) *_{n,p}^m \phi(u)$ and $\phi(v *_{m,p}^n w) \equiv \phi(w) *_{n,p}^m \phi(v)$ modulo $O'_{m,n}(V)$. Thus by Lemma 2.7 and the definition of $O_{m,n}(V)$,

$$
\begin{aligned}
&\phi((V *_{p_1}^n O_{p_1}(V)) *_{m,p_1}^n V) \subseteq V *_{n,p_1}^m \phi(V *_{p_1}^n O_{p_1}(V)) + O'_{m,n}(V) \\
&\subseteq V *_{n,p_1}^m (O_{p_1}(V) *_{n,p_1}^{p_1} V + O'_{p_1,n}(V)) + O_{m,n}(V) \\
&\subseteq (V *_{p_1}^m O_{p_1}(V)) *_{n,p_1}^m V + O_{m,n}(V) \\
&\subseteq O_{m,n}(V).
\end{aligned}
$$

That is, $\phi(O'''_{n,m}(V)) \subset O_{m,n}(V)$.

Finally we deal with $O''_{n,m}(V)$. For $u, a, b, c \in V$, by the discussion above, we have

$$
\begin{aligned}
&\phi[u *_{m,p_3}^n ((a *_{p_1,p_2}^{p_3} b) *_{m,p_1}^{p_3} c - a *_{m,p_2}^{p_3} (b *_{m,p_1}^{p_2} c))] \\
&\equiv \phi((a *_{p_1,p_2}^{p_3} b) *_{m,p_1}^{p_3} c - a *_{m,p_2}^{p_3} (b *_{m,p_1}^{p_2} c)) *_{n,p_3}^m \phi(u) \pmod{O'_{m,n}(V)} \\
&\equiv [\phi(c) *_{p_3,p_1}^m \phi(a *_{p_1,p_2}^{p_3} b) - \phi(b *_{m,p_1}^{p_2} c) *_{p_3,p_2}^m \phi(a) + x_1] *_{n,p_3}^m \phi(u),
\end{aligned}
$$

for some $x_1 \in O'_{m,p_3}(V)$. Since by Lemma 2.7, $O'_{m,p_3}(V) *_{n,p_3}^m V \subseteq O_{m,n}(V)$, we have

$$
\begin{aligned}
&\phi[u *_{m,p_3}^n ((a *_{p_1,p_2}^{p_3} b) *_{m,p_1}^{p_3} c - a *_{m,p_2}^{p_3} (b *_{m,p_1}^{p_2} c))] \\
&\equiv [\phi(c) *_{p_3,p_1}^m (\phi(b) *_{p_3,p_2}^{p_1} \phi(a) + x)] *_{n,p_3}^m \phi(u) \\
&\quad -[(\phi(c) *_{p_2,p_1}^m \phi(b) + y) *_{p_3,p_2}^m \phi(a)] *_{n,p_3}^m \phi(u)
\end{aligned}
$$

for some $x \in O'_{p_1,p_3}(V)$ and $y \in O'_{m,p_2}(V)$. By Lemma 2.7, we have

$$
(\phi(c) *_{p_3,p_1}^m O'_{p_1,p_3}(V)) *_{n,p_3}^m \phi(u) \subseteq O_{m,n}(V)
$$

and

$$
\begin{aligned}
&(O'_{m,p_2}(V) *_{p_3,p_2}^m \phi(a)) *_{n,p_3}^m \phi(u) \\
&\equiv O'_{m,p_2}(V) *_{n,p_2}^m (\phi(a) *_{n,p_3}^{p_2} \phi(u)) \pmod{O_{m,n}(V)} \\
&\subseteq O_{m,n}(V).
\end{aligned}
$$

Hence by Lemma 2.8, we have

$$
\begin{aligned}
&\phi[u*^{n}_{m,p_3}((a *^{p_3}_{p_1,p_2} b)*^{p_3}_{m,p_1} c - a*^{p_3}_{m,p_2}(b*^{p_2}_{m,p_1} c))]\\
\equiv\ &[\phi(c) *^{m}_{p_3,p_1} (\phi(b) *^{p_1}_{p_3,p_2} \phi(a)) - (\phi(c) *^{m}_{p_2,p_1} \phi(b)) *^{m}_{p_3,p_2} \phi(a)] *^{m}_{n,p_3} \phi(u)\\
\equiv\ &0 \pmod{O_{m,n}(V)}.
\end{aligned}
$$

Thus $\phi : A_{n,m}(V) \to A_{m,n}(V)$ is a well defined bimodule isomorphism. $\square$

It is proved in [DLM3] that the identity map on V induces epimorphism of associative algebras from $A_n(V)$ to $A_m(V)$, for $m \leq n$. A similar result holds here.

Proposition 3.3. *Let m, n, l be nonnegative integers such that $m-l, n-l$ are nonnegative. Then $A_{n-l,m-l}(V)$ is an $A_n(V)$-$A_m(V)$-bimodule and the identity map on V induces an epimorphism of $A_n(V)$-$A_m(V)$-bimodules from $A_{n,m}(V)$ to $A_{n-l,m-l}(V)$.*

Proof: It is good enough to prove the results for $l = 1$. We first show that $u *^{p_3}_{p_1,p_2} v \equiv u *^{p_3-1}_{p_1-1,p_2-1} v \mod O'_{p_3-1,p_1-1}(V)$, for $p_1, p_2, p_3 \in \mathbb{Z}_+$. Let u be homogeneous. Then

$$
\begin{aligned}
u *^{p_3}_{p_1,p_2} v =& \sum_{i=0}^{p_2} \binom{p_1+p_3-p_2+i}{i} (-1)^i \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}\, u+p_1-1}}{z^{p_1+p_3-p_2+i}}\\
&+ \sum_{i=0}^{p_2} \binom{p_1+p_3-p_2+i}{i} (-1)^i \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}\, u+p_1-1}}{z^{p_1+p_3-p_2+i+1}}\\
\equiv& \sum_{i=0}^{p_2-1} \binom{p_1+p_3-p_2+i}{i} (-1)^i \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}\, u+p_1-1}}{z^{p_1+p_3-p_2+i}}\\
&+ \sum_{i=0}^{p_2-2} \binom{p_1+p_3-p_2+i}{i} (-1)^i \mathrm{Res}_z Y(u,z) \frac{(1+z)^{\mathrm{wt}\, u+p_1-1}}{z^{p_1+p_3-p_2+i+1}} \qquad \pmod{O'_{p_3-1,p_1-1}(V)}\\
=& \left[\sum_{i=0}^{p_2-1} (-1)^i \binom{p_1+p_3-1-p_2+i}{i} + \sum_{i=1}^{p_2-1} (-1)^i \binom{p_1+p_3-1-p_2+i}{i-1} \right]\\
&\cdot \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt} u+p_1-1}}{z^{p_1+p_3-p_2+i}}\\
&+ \sum_{i=1}^{p_2-1} (-1)^{i-1} \binom{p_1+p_3-1-p_2+i}{i-1} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt} u+p_1-1}}{z^{p_1+p_3-p_2+i}}\\
=& u *^{p_3-1}_{p_1-1,p_2-1} v.
\end{aligned}
$$

In particular, we have $u *^{n}_{m} v \equiv u *^{n-1}_{m-1} v \mod O'_{n-1,m-1}(V)$ and $u \bar{*}^{n}_{m} v \equiv u \bar{*}^{n-1}_{m-1} v \mod O'_{n-1,m-1}(V)$. It remains to prove that $O_{n,m}(V) \subset O_{n-1,m-1}(V)$. Clearly, $O'_{n,m}(V) \subset O'_{n-1,m-1}(V)$ by Lemma 2.4. Recall from [DLM3] that $O_m(V) \subset O_{m-1}(V)$ for any m. Using the relation $u *^{p_3}_{p_1,p_2} v \equiv u *^{p_3-1}_{p_1-1,p_2-1} v \mod O'_{p_3-1,p_1-1}(V)$ and Lemma 2.7 and the definition of $O_{n,m}(V)$, one can easily show that $O''_{n,m}(V) \subset O_{n-1,m-1}(V)$, $(V *^{n}_{p_1} O_{p_1}(V)) *^{n}_{m,p_1} V \subset O_{n-1,m-1}(V)$. $\square$

We next study the tensor product $A_{n,p}(V)\otimes_{A_p(V)} A_{p,m}(V)$ which is an $A_n(V)$-$A_m(V)$-bimodule.

Proposition 3.4. *Define the linear map ψ: $A_{n,p}(V)\otimes_{A_p(V)} A_{p,m}(V)\to A_{n,m}(V)$ by*

$$\psi(u\otimes v)=u*_{m,p}^n v,$$

for $u\otimes v\in A_{n,p}(V)\otimes_{A_p(V)} A_{p,m}(V)$. Then ψ is an $A_n(V)-A_m(V)$- bimodule homomorphism from $A_{n,p}(V)\otimes_{A_p(V)} A_{p,m}(V)$ to $A_{n,m}(V)$.

Proof: First we prove that ψ is well defined. By Lemma 2.8, if $u\in O_{n,p}(V)$ or $v\in O_{p,m}(V)$, then $\psi(u\otimes v)=0$. For $u\in A_{n,p}(V), w\in A_p(V), v\in A_{p,m}(V)$, we have

$$\psi((u*_{p,p}^n w)\otimes v)=(u*_{p,p}^n w)*_{m,p}^n v$$

$$=u*_{m,p}^n (w*_{m,p}^p v)=\psi(u\otimes(w*_{m,p}^p v)).$$

Therefore ψ is well defined.

For $a\in A_n(V)$, $b\in A_m(V)$ and $u\otimes v\in A_{n,p}(V)\otimes_{A_p(V)} A_{p,m}(V)$, by the definition of $A_{n,m}(V)$, we have

$$a\bar{*}_m^n\psi(u\otimes v)=a\bar{*}_m^n(u*_{m,p}^n v)=(a\bar{*}_p^n u)*_{m,p}^n v=\psi((a\bar{*}_p^n u)\otimes v);$$

$$\psi(u\otimes v)*_m^n b=(u*_{m,p}^n v)*_m^n b=u*_{m,p}^n (v*_m^p b)=\psi(u\otimes(v*_m^p b)).$$

$\square$

It is worthy to point out that the map ψ is not surjective in general. For example, if $V=V^\natural$ is the moonshine vertex operator algebra constructed in [FLM] then V is rational (see [D], [DGH] and [M]) and $V_1=0$. Thus by Theorem 4.13 below, $A_{2,1}(V)=0$ and $A_{2,2}(V)\neq 0$. This shows that $A_{2,1}(V)\otimes_{A_1(V)}A_{1,2}(V)=0$ and $\psi: A_{2,1}(V)\otimes_{A_1(V)}A_{1,2}(V)\to A_2(V)$ is the zero map.

4 Connection to representation theory

Let $M=\bigoplus_{n\in\mathbb{Z}_+} M(n)$ be an admissible V-module such that $M(0)\neq 0$ (cf. [DLM2]). For $u\in V$, define $o_{n,m}(u): M(m)\mapsto M(n)$ by

$$o_{n,m}(u)w=u_{wtu+m-n-1}w,$$

for homogeneous $u\in V$ and $w\in M(m)$ where $u_{wtu+m-n-1}$ is the component operator of $Y_M(u,z)=\sum_{n\in\mathbb{Z}} u_n z^{-n-1}$.

The following lemma gives the representation theory reason for Proposition 3.4.

Lemma 4.1. *Let $a,b\in V$, $m,n,p\in\mathbb{Z}_+$ and $w\in M(m)$, then*

$$o_{n,m}(a*_{m,p}^n b)w=o_{n,p}(a)o_{p,m}(b)w.$$

Proof: We have the following computation on $M(m)$:

$$o_{n,m}(a *_{m,p}^{n} b) = o_{n,m}\left(\sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\mathrm{Res}_z\frac{(1+z)^{wta+m}}{z^{n+m-p+i+1}}Y(a,z)b\right)$$

$$= \sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{wta+m}\binom{wta+m}{j}(a_{j-n-m+p-i-1}b)_{wta+wtb-j+i+2m-p-1}$$

$$= \sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{wta+m}\binom{wta+m}{j}$$

$$\cdot\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}(z_1-z_2)^{j-n-m+p-i-1}z_2^{wta+wtb-j+i+2m-p-1}Y(Y(a,z_1-z_2)b,z_2)$$

$$= \sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}$$

$$\cdot\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}z_1^{wta+m}(z_1-z_2)^{-n-m+p-i-1}z_2^{wtb+i+m-p-1}Y(Y(a,z_1-z_2)b,z_2)$$

$$= \sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{\infty}\binom{-n-m+p-i-1}{j}$$

$$\cdot\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}z_1^{-n-m+p-i-1-j}(-z_2)^j z_1^{wta+m}z_2^{wtb+i+m-p-1}Y(a,z_1)Y(b,z_2)$$

$$-\sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{\infty}\binom{-n-m+p-i-1}{j}$$

$$\cdot\mathrm{Res}_{z_2}\mathrm{Res}_{z_1}(-z_2)^{-n-m+p-i-1-j}z_1^j z_1^{wta+m}z_2^{wtb+i+m-p-1}Y(b,z_2)Y(a,z_1)$$

$$= \sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{\infty}\binom{-n-m+p-i-1}{j}(-1)^j$$

$$\cdot a_{wta-n+p-i-j-1}b_{wtb+i+j+m-p-1}$$

$$-\sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{\infty}\binom{-n-m+p-i-1}{j}$$

$$\cdot(-1)^{n+m-p+j+i+1}b_{wtb-n-2-j}a_{wta+m+j}$$

$$= \sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{p}\binom{n+m-p+i}{i}\binom{-n-m+p-i-1}{k-i}$$

$$\cdot a_{wta-n+p-k-1}b_{wtb+m-p+k-1}$$

$$-\sum_{i=0}^{p}(-1)^i\binom{n+m-p+i}{i}\sum_{j=0}^{\infty}\binom{-n-m+p-i-1}{j}$$

$$\cdot(-1)^{n+m-p+j+i+1}b_{wtb-n-2-j}a_{wta+m+j}.$$

Note that $a_{wta+m+j} = b_{wtb+m-p+k-1} = 0$ on $M(m)$ for $j \geq 0$ and $k > p$. Also,

$$\sum_{i=0}^{k}\binom{n+m-p+i}{i}\binom{-n-m+p-i-1}{k-i} = 0,\ \ k = 1, 2, \cdots, p.$$

The proof is complete. $\square$

Corollary 4.2. *For $u, v \in V$ and $w \in M(m)$, we have*

$$o_{n,m}(u *_m^n v)w = o_{n,m}(u)o_{m,m}(v)w, \quad o_{n,m}(u \bar{*}_m^n v)w = o_{n,n}(u)o_{n,m}(v)w.$$

Let W be a weak V-module and $m \in \mathbb{Z}_+$. Following [DLM2] we define

$$\Omega_m(W) = \{w \in W | u_{wtu-1+k}w = 0, \text{for all homogeneous } u \in V \text{ and } k > m\}.$$

By Corollary 4.2 or [DLM3] we have:

Theorem 4.3. *Let W be a weak V-module. Then $\Omega_m(W)$ is an $A_m(V)$-module such that $a + O_m(V)$ acts as $o(a) = a_{wta-1}$ for homogeneous $a \in V$.*

We also have the following theorem from [DLM3].

Theorem 4.4. *Let $M = \bigoplus_{k=0}^{\infty} M(k)$ be an admissible V-module such that $M(0) \neq 0$. Then*

(1) $\bigoplus_{k=0}^{m} M(k)$ is an $A_m(V)$-submodule of $\Omega_m(M)$. Furthermore, if M is irreducible, $\Omega_m(M) = \bigoplus_{k=0}^{m} M(k)$.

(2) Each $M(k)$ is an $A_m(V)$-submodule for $k = 0, ..., m$. Moreover, if M is irreducible, each $M(k)$ is a simple $A_m(V)$-module.

We are now in a position to understand the representation theory meaning of $A_{n,m}(V)$. First observe that $\mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ is an $A_n(V)$-$A_m(V)$-bimodule such that $(a \cdot f \cdot b)(w) = af(bw)$ for $a \in A_n(V), b \in A_m(V)$, $f \in \mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ and $w \in M(m)$. Set $o_{n,m}(V) = \{o_{n,m}(v) | v \in V\}$.

Proposition 4.5. *The $o_{n,m}(V)$ is an $A_n(V)$-$A_m(V)$-subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ and $v \mapsto o_{n,m}(v)$ for $v \in V$ induces an $A_n(V)$-$A_m(V)$-bimodule epimorphism from $A_{n,m}(V)$ to $o_{n,m}(V)$.*

Proof: Clearly, $o_{n,m}(V)$ is an $A_n(V)$-$A_m(V)$-subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M(m), M(n))$ by Corollary 4.2. The same corollary also shows that the map $v \mapsto o_{n,m}(v)$ is an $A_n(V)$-$A_m(V)$-bimodule epimorphism if we can prove that $o_{n,m}(c) = 0$ for $c \in O_{n,m}(V)$. First let $c \in O'_{n,m}(V)$. If $c = L(-1)u + (L(0) + m - n)u$, $o_{n,m}(c) = 0$ is clear. If $c = u \circ_m^n v$, then by Lemma 2.2 we can assume that $c = a \bar{*}_m^n b$ for some $a \in O_n(V)$ and $b \in V$. Since $o(a) = 0$ we see from Corollary 4.2 that $o_{n,m}(c) = 0$. Next we assume that $c \in O''_{n,m}(V)$. Using Lemma 4.1 repeatedly shows that $o_{n,m}(c) = 0$. Finally take $c \in O'''_{n,m}(V)$. Again by Lemma 4.1 and Corollary 4.2, $o_{n,m}(c) = 0$, as desired. $\square$

It is proved in [DLM3] that for any given $A_m(V)$-module U which cannot factor through $A_{m-1}(V)$ there is a unique admissible V-module $\bar{M}(U) = \bigoplus_{k=0}^{\infty} \bar{M}(U)(k)$ of Verma type such that $\bar{M}(U)(m) = U$. The construction of $\bar{M}(U)$ is implicit in [DLM3]. We now recover this result and give an explicit construction of $\bar{M}(U)$ by using the bimodules $A_{n,m}(V)$. As a byproduct of this construction we can determine the structure of $A_{n,m}(V)$ explicitly if V is rational. We also expect that this new construction of $\bar{M}(U)$ will help our further study of representation theory of V.

We need the following result on the relation between $A_{n,n}(V)$ and $A_n(V)$:

Proposition 4.6. *For any $n \geq 0$, the $A_n(V)$ and $A_{n,n}(V)$ are the same.*

Proof: It is good enough to prove that any $A_n(V)$-module U is also an $A_{n,n}(V)$-module. Recall from the definition of $A_n(V)$ and $A_{n,n}(V)$ that $A_n(V) = V/O'_{n,n}(V)$ and $A_{n,n}(V) = V/O_{n,n}(V)$ where $O_{n,n}(V) = O'_{n,n}(V) + O''_{n,n}(V) + O'''_{n,n}(V)$. So we have to prove that $O''_{n,n}(V) + O'''_{n,n}(V)$ acts on U trivially. By Theorem 4.1 of [DLM3], there exists an admissible V-module $M = \bigoplus_{k=0}^{\infty} M(k)$ such that $M(n) = U$. We can not assume and do not need to assume that $M(0) \neq 0$. Note that the action of $A_n(V)$ on $U \subset M$ comes from the $A_n(V)$-module structure. This is, for $v \in A_n(V)$, $o(v) = o_{n,n}(v)$ is the module action of $A_n(V)$ on U. By Lemma 4.1 we immediately see that $O''_{n,n}(V) + O'''_{n,n}(V) = 0$ on U, as desired. $\square$

Let U be an $A_m(V)$-module which can not factor through $A_{m-1}(V)$. Set

$$M(U) = \bigoplus_{n \in \mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U.$$

Then $M(U)$ is naturally $\mathbb{Z}_+$-graded such that $M(U)(n) = A_{n,m}(V) \otimes_{A_m(V)} U$. By Proposition 4.6, $M(U)(m)$ and U are isomorphic $A_m(V)$-modules. The $M(U)$ will be proved to be the $\bar{M}(U)$ defined in [DLM3].

Recall Proposition 3.4. For $u \in V$, and $p, n \in \mathbb{Z}$, define an operator u_p from $M(U)(n)$ to $M(U)(n - wtu - p - 1)$ by

$$u_p(v \otimes w) = \begin{cases} (u *_{m,n}^{wtu-p-1+n} v) \otimes w, & \text{if } wtu - 1 - p + n \geq 0, \\ 0, & \text{if } wtu - 1 - p + n < 0, \end{cases}$$

for $v \in A_{n,m}(V)$ and $w \in U$. We need to prove that u_p is well defined. Let $v \in O_{n,m}(V)$ and $w \in U$. By Lemma 2.8, $u *_{m,n}^{wtu-p-1+n} v \in V *_{m,n}^{wtu-p-1+n} O_{n,m}(V) \subseteq O_{wtu-p-1+n,m}(V)$, so we have $u_p(v \otimes w) = 0$. Now let $a \in A_m(V)$, $v \in A_{n,m}(V)$, $w \in U$. Then

$$\begin{aligned} u_p((v *_m^n a) \otimes w) &= (u *_{m,n}^{wtu-p-1+n} (v *_m^n a)) \otimes w \\ &= ((u *_{m,n}^{wtu-p-1+n} v) *_m^{wtu-p-1+n} a) \otimes w \\ &= (u *_{m,n}^{wtu-p-1+n} v) \otimes a \cdot w = u_p(v \otimes a \cdot w). \end{aligned}$$

Thus u_p is well defined. Set

$$Y_M(u, z) = \sum_{p \in \mathbb{Z}} u_p z^{-p-1}.$$

Lemma 4.7. *For homogeneous $u \in V$, $v \otimes w \in A_{n,m}(V) \otimes_{A_m(V)} U$ and $p \in \mathbb{Z}$, we have*
(1) $u_p(v \otimes w) = 0$, for p sufficiently large;
(2) $Y_M(\mathbf{1}, z) = \mathrm{id}$.

Proof: (1) is clear and we only need to deal with (2). By the definition of u_p, we have

$$\begin{aligned} \mathbf{1}_p(v \otimes w) &= \sum_{i=0}^{n} (-1)^i \binom{-p-1+m+i}{i} \mathrm{Res}_z \frac{(1+z)^m}{z^{-p+m+i}} (Y(u, z)v) \otimes w \\ &= \sum_{i=0}^{n} (-1)^i \binom{-p-1+m+i}{i} \sum_{j=0}^{m} \binom{m}{j} (\mathbf{1}_{p-i-j} v) \otimes w \end{aligned}$$

Thus $\mathbf{1}_p(v\otimes w)=0$ if $p<-1$ and $\mathbf{1}_{-1}(v\otimes w)=v\otimes w$. Clearly, $\mathbf{1}_p(v\otimes w)=0$ if $p\geq n$. If $-1<p<n$, then

$$\begin{aligned}\mathbf{1}_p(v\otimes w)&=\sum_{i=0}^{p+1}(-1)^i\binom{m-p-1+i}{i}\binom{m}{p+1-i}v\otimes w\\&=\sum_{i=0}^{p+1}(-1)^{i+p+1}\binom{m}{p+1}\binom{p+1}{i}v\otimes w\\&=0.\end{aligned}$$

That is, $Y_M(\mathbf{1},z)=\mathrm{id}$. $\square$

We next have the commutator formula:

Lemma 4.8. *For $a,b\in V$, we have*

$$[Y_M(a,z_1),Y_M(b,z_2)]=\mathrm{Res}_{z_0}z_2^{-1}\delta(\frac{z_1-z_0}{z_2})Y_M(Y(a,z_0)b,z_2),$$

or equivalently, for $p,q\in\mathbb{Z}$

$$[a_p,b_q]=\sum_{i\geq 0}\binom{p}{i}(a_ib)_{p+q-i}.$$

Proof: Recall Lemma 2.3 and the definition of $A_{n,m}(V)$. For $p,q\in\mathbb{Z}$ and $v\otimes w\in A_{n,m}(V)\otimes_{A_m(V)}U$ we need to prove that

$$a_pb_q(v\otimes w)-b_qa_p(v\otimes w)=\sum_{i=0}^{\infty}\binom{p}{i}(a_ib)_{p+q-i}(v\otimes w).$$

It $wta+wtb-p-q-2+n<0$, this is clear from the definition of the actions. So we now assume that $wta+wtb-p-q-2+n\geq 0$. If $wta-p-1+n, wtb-q-1+n\geq 0$, then by Lemma 2.3 we have

$$\begin{aligned}&a_pb_q(v\otimes w)-b_qa_p(v\otimes w)\\&=a_p(b*_{m,n}^{wtb-q-1+n}v)\otimes w-b_q(a*_{m,n}^{wta-p-1+n}v)\otimes w\\&=\left(a*_{m,wtb-q-1+n}^{wta+wtb-p-q-2+n}(b*_{m,n}^{wtb-q-1+n}v)\right)\otimes w\\&\quad-\left(b*_{m,wta-p-1+n}^{wta+wtb-p-q-2+n}(a*_{m,n}^{wta-p-1+n}v)\right)\otimes w\\&=\left((a*_{n,wtb-q-1+n}^{wta+wtb-p-q-2+n}b)*_{m,n}^{wta+wtb-p-q-2+n}v\right)\otimes w\\&\quad-\left((b*_{n,wta-p-1+n}^{wta+wtb-p-q-2+n}a)*_{m,n}^{wta+wb-p-q-2+n}v\right)\otimes w\\&=\left((\mathrm{Res}_z(1+z)^pY_M(a,z)b)*_{m,n}^{wta+wtb-p-q-2+n}v\right)\otimes w\\&=(\sum_{i=0}^{\infty}\binom{p}{i}(a_ib)*_{m,n}^{wta+wtb-p-q-2+n}v)\otimes w\\&=\sum_{i=0}^{\infty}\binom{p}{i}(a_ib)_{p+q-i}(v\otimes w).\end{aligned}$$

It remains to prove the result for $wta-p-1+n \geq 0, wtb-q-1+n < 0$ or $wta-p-1+n < 0, wtb-q-1+n \geq 0$. If $wta-p-1+n < 0, wtb-q-1+n \geq 0$ then $b_q a_p(v \otimes w) = 0$ and

$$
\begin{aligned}
a_p b_q(v \otimes w) - b_q a_p(v \otimes w) &= \Big((a *_{n,wtb-q-1+n}^{wta+wtb-p-q-2+n} b) *_{m,n}^{wtp+wtb-p-q-2+n} v\Big) \otimes w \\
&= \big((\mathrm{Res}_z(1+z)^p Y_M(a,z)b) *_{m,n}^{wta+wtb-p-q-2+n} v\big) \otimes w \\
&= \sum_{i=0}^{\infty} \binom{p}{i} (a_i b)_{p+q-i}(v \otimes w)
\end{aligned}
$$

where we have used Lemma 4.9 below. Similarly, the result holds for $wta-p-1+n \geq 0, wtb-q-1+n < 0$. $\square$

Lemma 4.9. *Let $u, v \in V$ and $m \in \mathbb{Z}_+$, $p_1, p_2 \in \mathbb{Z}$ such that $p_1 + p_2 - m \geq 0$. If $p_1 \geq 0, p_2 < 0$, then*

$$u *_{m,p_1}^{p_1+p_2-m} v - \mathrm{Res}_z(1+z)^{wtu-1+m-p_2} Y(u,z)v \in O'_{p_1+p_2-m,m}(V)$$

and if $p_1 < 0, p_2 \geq 0$, then

$$-v *_{m,p_2}^{p_1+p_2-m} u - \mathrm{Res}_z(1+z)^{wtu-1+m-p_2} Y(u,z)v \in O'_{p_1+p_2-m,m}(V).$$

Proof: This lemma is similar to Lemma 2.3 where both p_1 and p_2 are nonnegative. If $p_2 < 0$ or $p_1 < 0$, then $v *_{m,p_2}^{p_1+p_2-m} u$ or $u *_{m,p_1}^{p_1+p_2-m} v$ is not defined. But we do need a version of Lemma 2.3 with either $p_1 < 0$ or $p_2 < 0$ in the proof of previous lemma.

First we assume that $p_1 \geq 0, p_2 < 0$. Then $-p_2 - 1 \geq 0$. From the definition, we have

$$u *_{m,p_1}^{p_1+p_2-m} v = \sum_{i=0}^{p_1} \binom{-p_2-1}{i} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{wtu+m}}{z^{p_2+i+1}}.$$

Since $\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{wtu+m}}{z^{p_2+i+1}} \in O'_{p_1+p_2-m,m}(V)$ if $i > p_1$ we see that

$$
\begin{aligned}
u *_{m,p_1}^{p_1+p_2-m} v &\equiv \sum_{i=0}^{-p_2-1} \binom{-p_2-1}{i} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{wtu+m}}{z^{p_2+i+1}}. \\
&= \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{wtu+m}}{z^{p_2+1}} (1+\frac{1}{z})^{-p_2-1} \\
&= \mathrm{Res}_z Y(u,z)v(1+z)^{wtu+m-p_2-1}.
\end{aligned}
$$

So in this case we have done.

If $p_1 < 0, p_2 \geq 0$ then the result in the first case gives

$$v *_{m,p_2}^{p_1+p_2-m} u \equiv \mathrm{Res}_z Y(v,z)u(1+z)^{wtv+m-p_1-1}$$

modulo $O'_{p_1+p_2-m,m}(V)$. Using the identity

$$Y(v,z)u \equiv (1+z)^{-wtu-wtv-2m+p_1+p_2} Y(u, \frac{-z}{1+z})v$$

modulo $O'_{p_1+p_2-m,m}(V)$ we see that

$$\begin{aligned}&\mathrm{Res}_z Y(v,z)u(1+z)^{wtv+m-p_1-1} \equiv \mathrm{Res}_z Y(u,\frac{-z}{1+z})v(1+z)^{-wtu-m+p_2-1}\\ &= -\mathrm{Res}_z Y(u,z)v(1+z)^{wtu+m-p_2-1}.\end{aligned}$$

The proof is complete. $\square$

Lemma 4.10. *For $i \in \mathbb{Z}_+$, we have*

$$\mathrm{Res}_{z_0} z_0^i (z_0+z_2)^{wta+n} Y_M(a, z_0+z_2) Y_M(b,z_2) = \mathrm{Res}_{z_0} z_0^i (z_2+z_0)^{wta+n} Y_M(Y(a,z_0)b, z_2)$$

on $M(U)(n)$ for $n \in \mathbb{Z}_+$.

Proof: Since $a_{wta+n} = 0$ on $M(U)(n)$, we have

$$\mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{wta+n} Y_M(b,z_2)Y_M(a,z_1)(v\otimes w) = 0.$$

Thus on $M(U)(n)$, we have

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^i (z_0+z_2)^{wta+n} Y_M(a, z_0+z_2) Y_M(b,z_2)\\ &= \mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{wta+n}(Y_M(a,z_1)Y_M(b,z_2) - Y_M(b,z_2)Y_M(a,z_1))\\ &= \mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{wta+n}[Y_M(a,z_1), Y_M(b,z_2)]\\ &= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}(z_1-z_2)^i z_1^{wta+n} z_2^{-1}\delta(\frac{z_1-z_0}{z_2}) Y_M(Y(a,z_0)b,z_2)\\ &= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^i (z_2+z_0)^{wta+n} z_1^{-1}\delta(\frac{z_2+z_0}{z_1}) Y_M(Y(a,z_0)b,z_2)\\ &= \mathrm{Res}_{z_0} z_0^i (z_2+z_0)^{wta+n} Y_M(Y(a,z_0)b,z_2),\end{aligned}$$

where we have used Lemma 4.8. $\square$

Lemma 4.11. *For $l \in \mathbb{Z}_+$, we have*

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^{-l}(z_2+z_0)^{wta+n} z_2^{wtb-n} Y_M(Y(a,z_0)b,z_2)\\ &= \mathrm{Res}_{z_0} z_0^{-l}(z_0+z_2)^{wta+n} z_2^{wtb-n} Y_M(a,z_0+z_2)Y_M(b,z_2)\end{aligned}$$

on $M(U)(n)$ for $n \geq 0$.

Proof: Take $v \otimes w \in A_{n,m}(V) \otimes_{A_m(V)} U = M(U)(n)$. Then

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^{-l}(z_2+z_0)^{wta+n} z_2^{wtb-n} Y_M(Y(a,z_0)b,z_2)(v\otimes w)\\ &= \sum_{j\in\mathbb{Z}_+}\binom{wta+n}{j} z_2^{wta+wtb-j} Y_M(a_{j-l}b, z_2)(v\otimes w)\\ &= \sum_{j\in\mathbb{Z}_+}\binom{wta+n}{j}\sum_{k\in\mathbb{Z}_+} z_2^{-l+k-n+1}(a_{j-l}b)_{wta+wtb-j+l-2-k+n}(v\otimes w)\\ &= \sum_{k\in\mathbb{Z}_+} z_2^{-l+k-n+1}\sum_{j\in\mathbb{Z}_+}\binom{wta+n}{j}\left((a_{j-l}b) *_{m,n}^k v\right)\otimes w\\ &= \sum_{k\in\mathbb{Z}_+} z_2^{-l+k-n+1}\left((\mathrm{Res}_z \frac{(1+z)^{wta+n}}{z^l} Y(a,z)b) *_{m,n}^k v\right)\otimes w\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-l}(z_0+z_2)^{wta+n} z_2^{wtb-n} Y_M(a, z_0+z_2) Y_M(b, z_2)(v\otimes w)\\
&= \sum_{i\in\mathbb{Z}_+} \binom{-l}{i} (-1)^i a_{wta+n-l-i} z_2^{wtb-n+i} Y_M(b, z_2)(v\otimes w)\\
&= \sum_{i\in\mathbb{Z}_+} \binom{-l}{i} (-1)^i a_{wta+n-l-i} \sum_{j\geq -n} z_2^{-n+i+j} b_{wtb-1-j}(v\otimes w)\\
&= \sum_{i\in\mathbb{Z}_+} \sum_{\substack{j\geq -n\\ l+i+j\geq 1}} \binom{-l}{i} (-1)^i z_2^{-n+i+j} \left(a *_{m,j+n}^{l+i+j-1} (b *_{m,n}^{j+n} v)\right) \otimes w\\
&= \sum_{k\in\mathbb{Z}_+} \sum_{j=-n}^{k+1-l} z_2^{-n+k+1-l} (-1)^{k+1-j-l} \binom{-l}{k+1-j-l} \left((a *_{n,j+n}^{k} b) *_{m,n}^{k} v\right) \otimes w\\
&= \sum_{k\in\mathbb{Z}_+} \sum_{j=0}^{k+1+n-l} z_2^{-n+k+1-l} (-1)^{k+1+n-j-l} \binom{-l}{k+1+n-j-l} \left((a *_{n,j}^{k} b) *_{m,n}^{k} v\right) \otimes w.
\end{aligned}
$$

So it is enough to prove that

$$
\sum_{j=0}^{k+1+n-l} (-1)^{k+1+n-j-l} \binom{-l}{k+1+n-j-l} a *_{n,j}^{k} b = \mathrm{Res}_z \frac{(1+z)^{wta+n}}{z^l} Y(a,z)b.
$$

But

$$
\begin{aligned}
&\sum_{j=0}^{k+1+n-l} (-1)^{k+1+n-j-l} \binom{-l}{k+1+n-j-l} a *_{n,j}^{k} b\\
&= \sum_{j=0}^{k+1+n-l} (-1)^{k+1+n-j-l} \binom{-l}{k+1+n-j-l} \sum_{i=0}^{j} (-1)^i \binom{k+n-j+i}{i}\\
&\qquad \mathrm{Res}_z \frac{(1+z)^{wta+n}}{z^{n+k-j+i+1}} Y(a,z)b\\
&= \sum_{j=0}^{k+1+n-l} (-1)^j \binom{-l}{j} \sum_{i=0}^{k+n+1-l-j} (-1)^i \binom{l+j+i-1}{i} \mathrm{Res}_z \frac{(1+z)^{wta+n}}{z^{l+j+i}} Y(a,z)b.
\end{aligned}
$$

By Proposition 5.3 in Appendix, we see that

$$
\sum_{j=0}^{k+1+n-l} (-1)^j \binom{-l}{j} \sum_{i=0}^{k+n+1-l-j} (-1)^i \binom{l+j+i-1}{i} \frac{1}{z^{l+j+i}} = \frac{1}{z^l}.
$$

This finishes the proof. $\square$

Corollary 4.12. *For $n \in \mathbb{Z}_+$, we have*

$$(z_2 + z_0)^{wta+n} Y_M(Y(a, z_0)b, z_2) = (z_0 + z_2)^{wta+n} Y_M(a, z_0 + z_2) Y_M(b, z_2)$$

on $M(U)(n)$.

Proof: The corollary is an immediate consequence of Lemmas 4.10 and 4.11. $\square$

We now can state another main theorem of this paper.

Theorem 4.13. *Let U be an $A_m(V)$-module which can not factor through $A_{m-1}(V)$, then $M(U) = \bigoplus_{n\in\mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$ is an admissible V-module with $M(U)(0) = A_{0,m}(V) \otimes_{A_m(V)} U \neq 0$ and with the following universal property: for any weak V-module W and any $A_m(V)$-morphism $\phi : U \to \Omega_m(W)$, there is a unique morphism $\bar{\phi} : M(U) \to W$ of weak V-modules which extends ϕ.*

Proof: Since $u \bar{*}_m^n 1 = u$, for $u \in V$ and $1 \in A_m(V)$, it follows that $M(U)$ is generated by $U \cong A_{m,m}(V) \otimes_{A_m(V)} U$ by Proposition 4.6. The fact that $M(U)$ is an admissible V-module follows from Lemmas 4.7-4.8 and Corollary 4.12. In fact, from the construction of $M(U)$ and Lemma 4.7, every condition in the definition of an admissible module (cf. [DLM2]) except the Jacobi identity is satisfied. It is well known that the Jacobi identity is equivalent to the commutator formula obtained in Lemma 4.7 and associativity in Corollary 4.12. If $M(U)(0) = A_{0,m}(V) \otimes_{A_m(V)} U = 0$, then U will be an $A_{m-1}(V)$-module, a contradiction. So $M(U)(0) \neq 0$.

It remains to prove the universal property of $M(U)$. Define the linear map: $\bar{\phi} : M(U) \to W$ by

$$\bar{\phi}(u \otimes w) = o_{n,m}(u)\phi(w),$$

for $u \in A_{n,m}(V)$ and $w \in U$. To be sure that $\bar{\phi}$ is well defined, we need to prove that

$$\bar{\phi}((u *_m^n v) \otimes w) = \bar{\phi}(u \otimes v \cdot w),$$

for $u \in A_{n,m}(V)$, $v \in A_m(V)$ and $w \in U$. Indeed, by Corollary 4.2 and the fact that ϕ is an $A_m(V)$-morphism, we have

$$\begin{aligned}&\bar{\phi}((u *_m^n v) \otimes w) = o_{n,m}(u *_m^n v)\phi(w) = o_{n,m}(u)o(v)\phi(w) \\ &= o_{n,m}(u)\phi(o(v)w) = \bar{\phi}(u \otimes o(v)w).\end{aligned}$$

Hence $\bar{\phi}$ is well defined.

For homogeneous $u \in V$, $v \otimes w \in A_{n,m}(V) \otimes_{A_m(V)} U$ and $p \in \mathbb{Z}$, by Lemma 4.1, we have

$$\begin{aligned}&\bar{\phi}(u_p(v \otimes w)) = \bar{\phi}((u *_{m,n}^{wtu-p-1+n} v) \otimes w) \\ &= o_{wtu-p-1+n,m}(u *_{m,n}^{wtu-p-1+n} v)\phi(w) \\ &= o_{wtu-p-1+n,n}(u)o_{n,m}(v)\phi(w) \\ &= o_{wtu-p-1+n,n}(u)\bar{\phi}(v \otimes w) \\ &= u_p\bar{\phi}(v \otimes w).\end{aligned}$$

This means that $\bar{\phi}$ is a morphism of weak V-modules. It is clear that $\bar{\phi}$ extends ϕ. $\square$

From the universal property of $M(U)$ we immediately have

Corollary 4.14. *The $M(U)$ is isomorphic to the admissible V-module $\bar{M}_m(U)$ constructed in [DLM3].*

Remark 4.15. *In the case $U = A_m(V)$, then $M(U) = \bigoplus_{n\geq 0} A_{n,m}(V)$ is an admissible V-module for any $m \geq 0$. On the other hand, it is easy to see that $\bigoplus_{n\geq 0} A_n(V)$ is* not *an admissible V-module. We certainly expect that the admissible module $\bigoplus_{n\geq 0} A_{n,m}(V)$ will play a significant role in our further study of representation theory for vertex operator algebras.*

Finally we study the $A_n(V)$-$A_m(V)$-bimodule structure of $A_{n,m}(V)$ if V is rational. The result below is not surprising from the representation theory point of view. Recall from [DLM3] that if V is rational then there are only finitely irreducible admissible V-modules up to isomorphisms and each irreducible admissible module is ordinary.

Theorem 4.16. *If V is a rational vertex operator algebra and $W^j = \bigoplus_{n\geq 0} W^j(n)$ with $W^j(0) \neq 0$ for $j = 1, 2, \cdots, s$ are all the inequivalent irreducible modules of V, then*

$$A_{n,m}(V) \cong \bigoplus_{l=0}^{\min\{m,n\}} \left(\bigoplus_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(W^i(m-l), W^i(n-l)) \right).$$

Proof: Since V is rational, $A_n(V)$ is isomorphic to the direct sum of full matrix algebras $\bigoplus_{i=1}^{s} \bigoplus_{k=0}^{n} \mathrm{End}_{\mathbb{C}}(W^i(k))$ by Theorem 4.10 of [DLM3]. So as an $A_n(V)$-$A_m(V)$-bimodule,

$$A_{n,m}(V) = \bigoplus_{i,j=1}^{s} \bigoplus_{0\leq p\leq m, 0\leq q\leq n} c_{i,j,p,q} \mathrm{Hom}_{\mathbb{C}}(W^i(p), W^j(q))$$

for some nonnegative integers $c_{i,j,p,q}$. So we need to prove that $c_{i,j,p,q} = 0$ if $i \neq j$ or $(p, q) \neq (m-l, n-l)$ for some $0 \leq l \leq \min\{m, n\}$ and $c_{i,i,m-l,n-l} = 1$.

We need a general result. Let U be an irreducible $A_p(V)$-module which is not an $A_{p-1}(V)$-module. Then $M(U) = \bigoplus_{k=0}^{\infty} A_{k,p}(V) \otimes_{A_p(V)} U$ is an admissible V-module generated by U by Theorem 4.13. Since V is rational, $M(U)$ is a direct sum of irreducible V-modules. Note that $M(U)(p) = U$ generates an irreducible submodule of $M(U)$. This shows that $M(U)$ is irreducible.

Consider irreducible admissible V-module $M = \bigoplus_{k=0}^{\infty} A_{k,m}(V) \otimes_{A_m(V)} W^i(m)$. Then M is isomorphic to W^i. Thus each $M(k)$ is isomorphic to $W(k)$ as $A_k(V)$-module. In particular, $A_{n,m}(V) \otimes_{A_m(V)} W^i(m)$ is isomorphic to $W^i(n)$. This shows that $c_{i,i,m,n} = 1$ and all other $c_{i,j,m,q} = 0$ if either $j \neq i$ or $q \neq n$.

Next we consider $M = \bigoplus_{k=0}^{\infty} A_{k,m}(V) \otimes_{A_m(V)} (W^i(m) + W^i(m-1))$. Then M is isomorphic to $W^i \bigoplus W^i$, $M(k) = W^i(k) \bigoplus W^i(k-1)$ for $k > 0$, and $M(0) = W^i(0)$. This implies that $c_{i,i,m-1,n-1} = 1$ and $c_{i,j,m-1,q} = 0$ if $j \neq i$ or $q \neq n-1$. Continuing in this way gives the result. $\square$

5 Appendix

In this section we present several identities involving formal variables used in the previous sections.

For $m,n\in\mathbb{Z}_+$, define

$$A_{m,n}(z)=\sum_{i=0}^{m}(-1)^i\binom{n+i}{i}\frac{(1+z)^{n+1}}{z^{n+i+1}}-\sum_{i=0}^{n}(-1)^m\binom{m+i}{i}\frac{(1+z)^i}{z^{m+i+1}}.$$

Proposition 5.1. *For all $m,n\in\mathbb{Z}_+$, $A_{m,n}(z)=1$.*

Proof: We will prove that $A_{m,n}(z)=A_{m,n+1}(z)$.

$$\begin{aligned}
A_{m,n+1}(z)&=\sum_{i=0}^{m}(-1)^i\binom{n+i+1}{i}\frac{(1+z)^{n+2}}{z^{n+i+2}}-\sum_{i=0}^{n+1}(-1)^m\binom{m+i}{i}\frac{(1+z)^i}{z^{m+i+1}}\\
&=\left[\sum_{i=0}^{m}(-1)^i\binom{n+i}{i}+\sum_{i=1}^{m}(-1)^i\binom{n+i}{i-1}\right]\left[\frac{(1+z)^{n+1}}{z^{n+i+2}}+\frac{(1+z)^{n+1}}{z^{n+i+1}}\right]\\
&-\sum_{i=0}^{n}(-1)^m\binom{m+i}{i}\frac{(1+z)^i}{z^{m+i+1}}-(-1)^m\binom{m+n+1}{n+1}\frac{(1+z)^{n+1}}{z^{m+n+2}}\\
&=A_{m,n}(z)+\left[\sum_{i=1}^{m}(-1)^i\binom{n+i}{i-1}+\sum_{i=0}^{m-1}(-1)^{i+1}\binom{n+i+1}{i}\right]\frac{(1+z)^{n+1}}{z^{n+i+2}}\\
&-(-1)^m\binom{m+n+1}{n+1}\frac{(1+z)^{n+1}}{z^{m+n+2}}+\sum_{i=0}^{m}(-1)^i\binom{n+i}{i}\frac{(1+z)^{n+1}}{z^{n+i+2}}\\
&=A_{m,n}(z)+\sum_{i=0}^{m}\left[(-1)^{i+1}\binom{n+i}{i}+(-1)^i\binom{n+i}{i}\right]\frac{(1+z)^{n+1}}{z^{n+i+2}}\\
&-\left[(-1)^m\binom{m+n+1}{n+1}+(-1)^{m+1}\binom{m+n+1}{n+1}\right]\frac{(1+z)^{n+1}}{z^{m+n+2}}\\
&=A_{m,n}(z).
\end{aligned}$$

Thus we have

$$A_{m,n}(z)=A_{m,0}(z)=\sum_{i=0}^{m}(-1)^i\frac{(1+z)}{z^{i+1}}-(-1)^m\frac{1}{z^{m+1}}$$

which is clearly equal to 1. $\square$

Proposition 5.2. *For $m,n\in\mathbb{Z}_+$, we have*

$$\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\left(\sum_{j=0}^{n-i}(-1)^j\binom{-m-i-1}{j}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{j+l}}{z_1^{j+i}}-\frac{1}{z_1^i}\right)=0. \tag{5.1}$$

Proof: Set

$$D_{n,m}(z_1,z_2)=\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\left(\sum_{j=0}^{n-i}(-1)^j\binom{-m-i-1}{j}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{j+l}}{z_1^{j+i}}-\frac{1}{z_1^i}\right).$$

We prove by induction on both m and n that $D_{n,m}(z_1, z_2) = 0$. It is clear that $D_{0,0}(z_1, z_2) = 0$. We now assume that $D_{n,m}(z_1, z_2) = 0$. Then

$$\begin{aligned}
D_{n,m+1}(z_1, z_2) &= \sum_{i=0}^{n}(-1)^i \binom{m+i+1}{i} \left(\sum_{j=0}^{n-i}(-1)^j \binom{-m-i-2}{j} \sum_{l=0}^{i} \binom{i}{l} \frac{z_2^{j+l}}{z_1^{j+i}} - \frac{1}{z_1^i} \right) \\
&= \sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \sum_{j=0}^{n-i}(-1)^j \binom{-m-i-1}{i} \frac{m+i+j+1}{m+1} \sum_{l=0}^{i} \binom{i}{l} \frac{z_2^{j+l}}{z_1^{j+i}} \\
&\quad - \sum_{i=0}^{n}(-1)^i \left[\binom{m+i}{i} + \binom{m+i}{i-1} \right] \frac{1}{z_1^i} \\
&= D_{n,m}(z_1, z_2) - \sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \frac{i}{m+1} \frac{1}{z_1^i} \\
&\quad + \sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \sum_{j=0}^{n-i}(-1)^j \binom{-m-i-1}{j} \frac{i+j}{m+1} \sum_{l=0}^{i} \binom{i}{l} \frac{z_2^{j+l}}{z_1^{j+i}}.
\end{aligned}$$

By induction assumption, we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \sum_{j=0}^{n-i}(-1)^j \binom{-m-i-1}{j} \frac{i+j}{m+1} \sum_{l=0}^{i} \binom{i}{l} \frac{z_2^{j+l}}{z_1^{j+i}} \\
&\quad - \sum_{i=0}^{n}(-1)^i \binom{m+i}{i} \frac{i}{m+1} \frac{1}{z_1^i} \\
&= 0.
\end{aligned}$$

Thus $D_{n,m+1}(z_1,z_2)=0$. We now deal with $D_{n+1,m}(z_1,z_2)$. We have the computation:

$$
\begin{aligned}
&D_{n+1,m}(z_1,z_2)\\
&=\sum_{i=0}^{n+1}(-1)^i\binom{m+i}{i}\left(\sum_{j=0}^{n+1-i}(-1)^j\binom{-m-i-1}{j}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{j+l}}{z_1^{j+i}}-\frac{1}{z_1^i}\right)\\
&=\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\sum_{j=0}^{n+1-i}(-1)^j\binom{-m-i-1}{j}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{j+l}}{z_1^{j+i}}\\
&\quad+(-1)^{n+1}\binom{m+n+1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{z_2^{l}}{z_1^{n+1}}\\
&\quad-\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\frac{1}{z_1^i}-(-1)^{n+1}\binom{m+n+1}{n+1}\frac{1}{z_1^{n+1}}\\
&=\sum_{i=0}^{n}(-1)^i\binom{m+i}{i}\binom{-m-i-1}{n+1-i}(-1)^{n+1-i}\sum_{l=0}^{i}\binom{i}{l}\frac{z_2^{n+1-i+l}}{z_1^{n+1}}\\
&\quad+(-1)^{n+1}\binom{m+n+1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{z_2^{l}}{z_1^{n+1}}\\
&\quad-(-1)^{n+1}\binom{m+n+1}{n+1}\frac{1}{z_1^{n+1}}+D_{n,m}(z_1,z_2)\\
&=\sum_{l'=1}^{n+1}\sum_{i=0}^{n}(-1)^{n+1}\binom{m+i}{i}\binom{-m-i-1}{n+1-i}\binom{i}{l'-n-1+i}\frac{z_2^{l'}}{z_1^{n+1}}\\
&\quad+\sum_{l'=1}^{n+1}(-1)^{n+1}\binom{m+n+1}{n+1}\binom{n+1}{l'}\frac{z_2^{l'}}{z_1^{n+1}}\\
&=\sum_{l'=1}^{n+1}\sum_{i=0}^{n}(-1)^{i}\binom{m+n+1}{n+1}\binom{n+1}{l'}\binom{l'}{n+1-i}\frac{z_2^{l'}}{z_1^{n+1}}\\
&\quad+\sum_{l'=1}^{n+1}(-1)^{n+1}\binom{m+n+1}{n+1}\binom{n+1}{l'}\frac{z_2^{l'}}{z_1^{n+1}}\\
&=0.
\end{aligned}
$$

The Proposition is proved. $\square$

Proposition 5.3. *Let $k,l\in\mathbb{Z}_+$ be such that $k+1-l\geq 0$ and $l\geq 1$, then*

$$\sum_{j=0}^{k+1-l}(-1)^j\binom{-l}{j}\sum_{i=0}^{k+1-l-j}(-1)^i\binom{l+i+j-1}{i}\frac{1}{z^{i+j+l}}=\frac{1}{z^l}.$$

Proof: We prove the result by induction on $l \geq 1$. For $l = 1$, the identity becomes

$$\sum_{j=0}^{k} \sum_{i=0}^{k-j} (-1)^i \binom{j+i}{i} \frac{1}{z^{j+i+1}} = \frac{1}{z}.$$

It is easy to deduce the above formula by induction on $k \geq 0$. Now, suppose the identity is true for l. Then

$$\frac{d}{dz} \left(\sum_{j=0}^{k+1-l} (-1)^j \binom{-l}{j} \sum_{i=0}^{k+1-l-j} (-1)^i \binom{l+i+j-1}{i} \frac{1}{z^{i+j+l}} \right) = \frac{d}{dz} \left(\frac{1}{z^l} \right),$$

whose left side is

$$\begin{aligned}
&\sum_{j=0}^{k+1-l} (-1)^j \binom{-l}{j} \sum_{i=0}^{k+1-l-j} (-1)^i \binom{l+i+j-1}{i} (-l-j-i) \frac{1}{z^{i+j+l+1}} \\
&= (-l) \sum_{j=0}^{k+1-l} (-1)^j \binom{-l-1}{j} \sum_{i=0}^{k+1-l-j} (-1)^i \binom{l+i+j}{i} \frac{1}{z^{i+j+l+1}} \\
&= (-l) \sum_{j=0}^{k-l} (-1)^j \binom{-l-1}{j} \sum_{i=0}^{k+1-l-j} (-1)^i \binom{l+i+j}{i} \frac{1}{z^{i+j+l+1}} \\
&\quad -l \binom{-l-1}{k+1-l} (-1)^{k+1-l} \frac{1}{z^{k+2}} \\
&= (-l) \sum_{j=0}^{k-l} (-1)^j \binom{-l-1}{j} \sum_{i=0}^{k-l-j} (-1)^i \binom{l+i+j}{i} \frac{1}{z^{i+j+l+1}} \\
&\quad -l \sum_{j=0}^{k-l} \binom{-l-1}{j} (-1)^{k+1-l} \binom{k+1}{k+1-l-j} \frac{1}{z^{k+2}} \\
&\quad -l \binom{-l-1}{k+1-l} (-1)^{k+1-l} \frac{1}{z^{k+2}} \\
&= (-l) \sum_{j=0}^{k-l} (-1)^j \binom{-l-1}{j} \sum_{i=0}^{k-l-j} (-1)^i \binom{l+i+j}{i} \frac{1}{z^{i+j+l+1}} \\
&\quad -l \binom{k+1}{l} (-1)^{k+1-l} \sum_{j=0}^{k-l} \binom{k+1-l}{j} (-1)^j \frac{1}{z^{k+2}} \\
&\quad -l \binom{k+1}{l} \frac{1}{z^{k+2}} \\
&= (-l) \sum_{j=0}^{k-l} (-1)^j \binom{-l-1}{j} \sum_{i=0}^{k-l-j} (-1)^i \binom{l+i+j}{i} \frac{1}{z^{i+j+l+1}},
\end{aligned}$$

where in the last step we have used the assumption that $k+1-(l+1) \geq 0$. So the identity holds for $l+1$. $\square$

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